A sharp Wigner-negativity bound for a single optical qubit

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Abstract

We prove Gagatsos's conjectured order-one-half Wigner–Husimi inequality for every state of one optical mode supported on zero and one photon, including arbitrary coherences and mixed states. A common noncentral Gaussian radius represents both sides for a pure state. The large-displacement argument uses the deficit in Gaussian Poincaré, and a finite rational coefficient certificate handles small displacement.

Theorem 1. *Let ρ be supported on* $\mathrm{span}\{|0\rangle, |1\rangle\}$. *With $[x, p] = i$ and $Q = \langle\alpha|\rho|\alpha\rangle/(2\pi)$, $\alpha = (x + ip)/\sqrt{2}$, one has*

$$\int |W|\, dx\, dp \;\le\; H(\rho) := \frac{1}{8\pi}\left(\int \sqrt{Q}\, dx\, dp\right)^2.$$

Equality holds exactly for vacuum.

This is Conjecture 1 of Gagatsos [1] on the optical single-rail qubit. The Wigner negativity of this subspace has been studied analytically [2]. Its pure states include coherent superpositions with a negative Wigner region, even arbitrarily close to vacuum.

1 One random radius

Apart from vacuum, rotation and an overall phase put a pure state in the form $(b|0\rangle + |1\rangle)/\sqrt{1+b^2}$, $b \ge 0$. In the complex α plane with measure $d^2\alpha$, the normalized Husimi density is $e^{-|\alpha|^2}|b+\alpha|^2/[\pi(1+b^2)]$ and the normalized Wigner density is $2e^{-2|\alpha|^2}(|b+2\alpha|^2-1)/[\pi(1+b^2)]$. These follow by taking coherent-state overlaps and the displaced-parity matrix elements in the zero–one basis. The Jacobian $dx\, dp = 2d^2\alpha$ makes the prefactor of the squared square-root integral $1/(4\pi)$ in these complex coordinates.

Let X, Y be independent standard normal variables, $R = ((b+X)^2 + Y^2)^{1/2}$, $t = b^2/2$, $m = \mathbb{E}R$ and $d = \mathbb{E}(1-R^2)_+$. The variable 2α in the vacuum Wigner density and the variable α in the normalized square root of the vacuum Husimi density have the same Gaussian law. Since $\mathbb{E}R^2 = b^2 + 2$, this gives

$$(1+b^2)H = m^2, \qquad (1+b^2)\|W\|_1 = \mathbb{E}|R^2 - 1| = 1 + 2t + 2d.$$

Thus the desired strict inequality is $m^2 > 1 + 2t + 2d$.

In polar coordinates the density of $S = R^2/2$ is $e^{-t-s}\sum_{k\ge0}(ts)^k/(k!)^2$ on $s > 0$ (expand the angular integral of $e^{2\sqrt{ts}\cos\theta}$). Equivalently, conditional on $K \sim \mathrm{Poisson}(t)$, S has gamma shape $K+1$ and unit scale. With $(a)_k = \Gamma(a+k)/\Gamma(a)$,

$$m = \sqrt{\pi/2}\, e^{-t} P(t), \qquad P(t) = \sum_{k\ge0} \frac{(3/2)_k}{(k!)^2} t^k. \tag{1}$$

Dropping $e^{-s} \le 1$ on $0 < s < 1/2$ bounds $\mathbb{E}[(1-2S)_+ \mid K = k]$ by $1/[2^{k+1}(k+2)!]$. Successive bounds have ratio $1/[2(k+3)] \le 1/6$. Averaging gives

$$d \le \tfrac{1}{4} e^{-5t/6}. \tag{2}$$

2 A uniform estimate for $t \ge 2$

For a Gaussian Sobolev function f,

$$2\,\mathrm{Var}(f) \le \mathbb{E}|\nabla f|^2 + |\mathbb{E}\nabla f|^2.$$

Indeed, in the orthonormal multivariate Hermite expansion the variance is the sum of squared nonconstant coefficients. The gradient energy weights each by its total degree, and the degree-one sum is $|\mathbb{E}\nabla f|^2$ by Gaussian integration by parts. This proves the inequality for polynomials and then by Gaussian Sobolev closure for the Lipschitz radius.

Apply it to $f = R$, whose gradient has norm one almost everywhere. The mean of the Y-component vanishes by symmetry. Cauchy–Schwarz applied to the other component yields

$$1 - \operatorname{Var} R \geq \tfrac{1}{2}(1 - |\mathbb{E}\nabla R|^2) \geq \tfrac{1}{2}\mathbb{E}[Y^2/R^2] = \frac{1 - e^{-t}}{4t}.$$

For the last equality, integrate $\mathbb{E}[Y^2 e^{-sR^2}] = (1+2s)^{-2} e^{-b^2 s/(1+2s)}$ over $s \geq 0$, using $v = s/(1+2s)$. For $t \geq 2$, $2te^{-5t/6}$ decreases and $1 - e^{-t}$ increases. At $t = 2$ the former is below $4/5$ and the latter above it (the exponential series gives $e^{5/3} > 5$, $e^2 > 5$). Hence $1 - \operatorname{Var} R > \tfrac{1}{2}e^{-5t/6} \geq 2d$. Substituting $\operatorname{Var} R = 2 + 2t - m^2$ proves the claim.

3 An exact finite certificate for $0 \leq t \leq 2$

By (1) and (2) it suffices to prove

$$\tfrac{157}{100} P(t)^2 > (1 + 2t)e^{2t} + \tfrac{1}{2}e^{7t/6}, \qquad \pi > 3.14.$$

For a reproducible integer check, set $v_j = (2j+1)\binom{2j}{j}$ and $C_k = \sum_{j=0}^{k} \binom{k}{j} v_j v_{k-j}$. The coefficient of P^2 at degree k is $C_k/(4^k k!)$. For $k = 0, \ldots, 16$ the following integer comparisons all hold:

$$157\, 6^k C_k > [100(k+1) + 1]48^k + 50\, 28^k.$$

Equivalently the degree-k surplus exceeds $2^k/(100k!)$. At degree zero the surplus is exactly $7/100$. These seventeen comparisons can be checked by Pascal's rule; the accompanying Lean file evaluates them in the kernel. The right-hand tail at $t \leq 2$ is bounded by $\sum_{k \geq 17}(k + 3/2)4^k/k!$. Consecutive terms have ratio below $1/4$, and $(4/3)(37/2)4^{17}/17! < 1/100$. Every omitted coefficient on the left is nonnegative. This proves strict inequality on the closed interval, including the one-photon endpoint $b = 0$.

Mixed states and equality. In a pure-state decomposition the Wigner norm is convex and H concave by the integral triangle inequality on the vector of square-root Husimi densities weighted by square-root probabilities [1]. Every nonvacuum state has a nonvacuum pure component with positive weight. The bound is strict there, while vacuum gives $\|W\|_1 = H = 1$. This finishes the proof. A common displacement translates both integrals, giving the same result for every displacement of this class.

AI contributions and verification. GPT-6.2 Astra (high) helped draft the account of the finite certificate. Lean checks seventeen coefficient comparisons and a tail inequality. Extending the result to every state on the zero–one-photon subspace also requires the Gaussian reduction, the estimate for $t \geq 2$ and the convexity step for mixtures. Those steps are written arguments. Lean arithmetic certificate.

References

[1] C. N. Gagatsos, *Wigner negativity costs phase-space area*, Conjecture 1 and Proposition 1 (2026). arXiv:2611.30969v1.

[2] K. M. Fonseca Romero, *Analytic Wigner Negativity for Mixtures and Superpositions of the Lowest Fock States*, Int. J. Theor. Phys. (2026). doi:10.1007/s10773-025-06241-w.