Computational-basis optimality of nonlocal stabilizer entropy for rank-seven spectra and two exceptional amplitudes

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26 November 2026

Abstract

We prove computational-basis optimality for every rank-seven bipartite pure state on two three-qubit registers. In arbitrary supporting register sizes, we give three full-dimensional rank-seven spectral regions with exact rational certificates. For equal registers of any local dimension $d = 2^n$, we prove optimality when at least $d - 2$ of the full d Schmidt amplitudes, counting zeros, coincide. The latter proof uses a two-particle occupation bound and gives an exact rank-15 example. These results use Sierant's analytic reduction of the local-unitary optimization. The unrestricted conjecture remains open.

1 Statement and reduction

Let $\mu_1 \geq \cdots \geq \mu_7 > 0$, $\sum_i \mu_i^2 = 1$, and put $t_i = \mu_i^2$. Coordinates start at zero, so μ_i is assigned to label $i - 1$. On two three-qubit registers define

$$|\psi_0\rangle = \sum_{i=1}^{7} \mu_i |i-1\rangle_A |i-1\rangle_B, \qquad \mathcal{P}(\psi) = 2^{-6} \sum_{P \in \{I,X,Y,Z\}^{\otimes 6}} \langle\psi|P|\psi\rangle^4.$$

The nonlocal stabilizer 2-Rényi entropy is $-\log_2$ of the maximum of $\mathcal{P}$ under separate unitaries on the registers. Conjecture 1 of Sierant [1] asks whether $|\psi_0\rangle$ is always optimal. Earlier formulations appear in [2,3]. His Theorems 1 and 2 cover dyadic-staircase spectra and ranks at most six, respectively.

Theorem. For every rank-seven state on three qubits per party, the maximum is attained at $|\psi_0\rangle$. With A ranging over four-element subsets of $\{1, \ldots, 7\}$, its value is

$$7\Big(\sum_i t_i^2\Big)^2 - 6\sum_i t_i^4 + 168 \sum_{\substack{|A|=4 \\ \bigoplus_{i\in A}(i-1)=0}} \prod_{i\in A} t_i. \tag{1}$$

Here $\oplus$ is binary XOR. The same conclusion holds for arbitrary qubit register sizes supporting rank seven if $\mu_1 = \mu_2 = \mu_3$ or $\mu_2 = \mu_3 = \mu_4$.

Reduction. First take equal local dimensions $d = 2^n$. Let Π be the antisymmetric stabilizer projector of [1, Eq. 39] on $\Lambda^4\mathbb{C}^d$, and put $q = 4/d$. For an orthonormal seven-frame u let u_A be the basis wedge in increasing order, normalized to one, and put $x_A = \|\Pi u_A\|^2/q$. If N has eigenvalues μ_i on this frame and is zero on its complement, the overlap in [1] is $c_{\mathrm{F}}(N^2) = q\sum_A x_A \prod_{i\in A} t_i$. In the coordinate frame x_A is one on affine planes and zero elsewhere. Denote that vector by x^0. It suffices to establish

$$\sum_A x_A \prod_{i\in A} t_i \;\leq\; \sum_A x_A^0 \prod_{i\in A} t_i. \tag{2}$$

Indeed, Lemma 1 of [1] reduces the two-unitary maximum to a single unitary. Proposition 2 and Theorem 7 there give $G \leq \mathcal{I} + 42d\, c_{\mathrm{F}}(N^2)$, with equality at the sorted computational frame on the first seven labels. The spectral term $\mathcal{I}$ is the first two terms of (1), and $42dq = 168$. Padding the smaller register with product qubits extends every dimension-independent statement to unequal registers by [1, Sec. III.1].

2 Rows valid for physical frames

All x_A are nonnegative. The certificate uses the following inequalities. The sets and distinguished indices in each row can be chosen arbitrarily in the seven-frame.

Index set	Sum being bounded	Upper bound
$\lvert T\rvert = 3$	$\sum_{A\supseteq T} x_A$	1
$\lvert Y\rvert = 5$	$\sum_{A\subseteq Y} x_A$	1
$\lvert Y\rvert = 6$	$\sum_{A\subseteq Y} x_A$	3
$\lvert Y\rvert = 6,\ i \in Y$	$\sum_{i\in A\subseteq Y} x_A$	2
$\lvert T\rvert = 2,\ d = 8$	$\sum_{A\supseteq T} x_A$	2

The five-site row is [1, Corollary 4]. Summing the six five-site rows inside a six-set counts each four-set twice and gives the six-site row. The distinguished six-site row is the last inequality in [1, Proposition 10], where edges label complementary four-sets of the six-frame.

For the triple and pair rows, write the coordinate projector as one rank-one block per plane direction. Its normalized vector has amplitude $\sqrt{q}$ on each parallel plane. Two distinct parallel planes have no common point. In the unnormalized k-particle occupation matrix of Π, an off-diagonal term from two such wedges requires $4-k$ common spectator indices. Thus for $k = 2, 3$ only diagonal plane terms survive. Each coordinate triple extends to exactly one affine plane. Each coordinate pair lies in $(d-2)/2$ planes, since each third point determines the fourth. Consequently the three-particle occupation matrix equals qI, and the two-particle one equals $q(d-2)I/2$, with the convention that a basis wedge contributes one to each diagonal entry indexed by a contained k-set. Exterior powers transform unitarily under basis changes, so both identities hold in every orthonormal basis.

Complete the frame to an orthonormal basis. For a fixed triple, the sum of the unscaled four-wedge weights in the complete basis is q. Keeping only frame indices gives the first row. When $d = 8$ there is exactly one omitted basis vector h. For a fixed pair the complete sum is $3q$, and the contribution containing h is exactly the complete triple sum q. The frame pair sum therefore equals $2q$. This subtraction uses the codimension-one hypothesis.

3 Exact coefficient certificate

Set $t_8 = 0$ and write $g_j = t_j - t_{j+1} \geq 0$, so $t_i = \sum_{j\geq i} g_j$. Expand (2) in the g_j. There are $\binom{10}{4} = 210$ quartic monomials. Given $1 \leq j_1 \leq \cdots \leq j_4 \leq 7$, define w_A as the number of orderings $(i_1, \ldots, i_4)$ of A with $i_\ell \leq j_\ell$ at all four positions. This is the coefficient of $\prod_\ell g_{j_\ell}$ multiplied by the product of the multiplicity factorials, common to both sides. The required inequalities are $w^{\mathsf{T}}x \leq w^{\mathsf{T}}x^0$.

Let $Rx \leq b$ denote the rows in the table. For each w the deposited certificate supplies a nonnegative rational vector y satisfying

$$R^{\mathsf{T}}y \geq w, \qquad b^{\mathsf{T}}y = w^{\mathsf{T}}x^0.$$

Multiplying the physical rows by y and adding proves the coefficient inequality. Summing with the nonnegative monomial weights proves (2) for the entire continuum of spectra. The accompanying standard-library Python checker reconstructs R, b, w, x^0, exhausts the sorted quadruples, and tests both relations in exact arithmetic. Linear programming was used to find the multipliers. Replay only needs the rational witnesses. A second implementation checks all 210 integer-scaled inequalities in the Lean 4.19.0 kernel. It counts admissible assignments by filling slots from narrowest to widest, multiplying the number of still-available labels at each step.

Only five witnesses use the pair row:

$$(1,2,7,7),\ (1,3,6,7),\ (1,3,7,7),\ (2,2,7,7),\ (2,3,7,7).$$

Every one contains a member of $\{1,2\}$ and a member of $\{2,3\}$. Under either degeneracy in the theorem all five monomials vanish, and the other 205 certificates establish (2) in every dimension. This completes the proof.

A check by hand. For flat rank seven, sum the six-site inequality over the seven possible omitted indices. Every four-set survives three times, so $\sum_A x_A \le 7$. Exactly seven affine planes occur among the first seven labels, attaining equality. This proves flat optimality for every supporting register size. Substituting $t_i = 1/7$ in (1) gives $211/343$ and hence a minimum entropy of $\log_2(343/211)$ bits. Independently, a direct sum over all 4,096 six-qubit Paulis checks this value and two unequal spectra. The unequal amplitudes proportional to $(7,6,5,4,3,2,1)$ give $134539/214375$.

4 Rank seven in larger registers

The following regions have interior among rank-seven spectra. For arbitrary supporting qubit registers, (1) gives the optimum whenever at least one of these conditions holds:

$$t_1 \le 3t_7, \qquad 2t_2 \le 3t_7, \qquad t_4 - t_5 \ge t_7. \tag{3}$$

These are conditions on the Schmidt *probabilities*. Unlike the degeneracy corollaries, they permit seven different positive entries.

Proof. Denote the vector of seven ones by $\mathbf{1}$ and the first-j indicator by E_j. In each of the three regions, $t = \sum_{j=1}^{7} h_j v_j$ for nonnegative h_j. In the first region take $v_j = \mathbf{1} + 2E_j$ for $j < 7$, $v_7 = \mathbf{1}$, $h_j = g_j/2$ for $j < 7$ and $h_7 = (3t_7 - t_1)/2$. In the second take $v_1 = E_1$, $v_j = 2\mathbf{1} + E_j$ for $2 \le j < 7$ and $v_7 = \mathbf{1}$. Here $h_j = g_j$ for $j < 7$ and $h_7 = 3t_7 - 2t_2$. In the third take $v_j = E_j$ for $j < 7$ and $v_7 = \mathbf{1} + E_4$. Here $h_j = g_j$ except $h_4 = g_4 - t_7$. All three substitutions recover each coordinate t_i.

For each ordered multiset of four ray indices $j_1, \dots, j_4$, use the weight

$$w_A = \sum_{\text{orderings } (i_1,\dots,i_4) \text{ of } A} \prod_{\ell=1}^{4} (v_{j_\ell})_{i_\ell}.$$

The earlier dual test applies unchanged to these weights. For each region its 210 deposited nonnegative rational multipliers (one vector per monomial) satisfy the two dual relations using only the first four row types of the table. These 630 certificates are checked exactly by `regions.py`; a separate implementation reconstructed the polynomial coefficients by multiplication and checked the certificates. This proves (2) throughout (3). The 630 new checks use rational arithmetic. The earlier Lean coverage is the 210 original inequalities.

Limit of this method. The factor three in the first condition is sharp for the four dimension-independent row types used here. Assign a four-set weight $1/5$, $1/8$ or $1/4$ according as it contains zero, one or two of the leading two indices. These nonnegative values satisfy every such row, as the checker verifies. For unnormalized $t = (R, R, 1, 1, 1, 1, 1)$ their objective exceeds the coordinate objective by $R(R-3)/2$ when $R > 3$. Normalization multiplies both objectives by the same positive factor. This is an obstruction to the linear relaxation. Physical realizability of these weights is a separate question.

5 Two exceptional Schmidt amplitudes at arbitrary rank

Let the two registers have equal local dimension $d = 2^n$. Suppose the entire list of d Schmidt amplitudes consists of $d-2$ copies of a and two entries b, c. The values are nonnegative and $(d-2)a^2 + b^2 + c^2 = 1$. The sorted computational-basis state attains the global optimum for every such spectrum, including

zeros and exceptions below the common value. For $d \le 4$ the source's rank theorem suffices. Consider $d \ge 8$.

Let u, v be the orthogonal exceptional eigenvectors of N. On four-wedges write n_u, n_v for the commuting projections testing occupancy of the two modes. Matching the four possible occupancy eigenvalues gives

$$\Lambda^4(N) = a^4 I + a^3(b-a)n_u + a^3(c-a)n_v + B\, n_u n_v, \qquad B = a^2(b-a)(c-a).$$

By the single-mode identity of [1, Appendix C, Theorem 17], $\Pi n_u \Pi = \Pi n_v \Pi = q\Pi$. Therefore the compression is $A\Pi + BK$ where

$$A = a^4 + qa^3(b+c-2a), \qquad K = \Pi n_u n_v \Pi, \qquad 0 \le K \le q\Pi.$$

Here $K \le \Pi n_u \Pi = q\Pi$ because the commuting occupation projections satisfy $n_u n_v \le n_u$. The scalar two-particle occupation matrix gives $T = \operatorname{Tr} K = q(d-2)/2$ independently of the pair. Thus $\operatorname{Tr}(K^2) \le qT$. For two coordinate vectors, each affine direction has at most one plane containing both vectors. The disjoint direction blocks in Eq. 39 of [1] then give eigenvalues zero and q for K, attaining this bound. If $R = (d-1)(d-2)/6$ is the number of directions, the squared Hilbert–Schmidt norm is

$$Q = RA^2 + 2ABT + B^2 \operatorname{Tr}(K^2).$$

The middle term is constant even when the exceptions straddle a and $B < 0$. The only frame-dependent term has coefficient $B^2 \ge 0$. The trace overlap for N^2 is constant, equal to $RA(a^2, b^2, c^2) + TB(a^2, b^2, c^2)$ using the same coefficient formulas. The coordinate pair therefore maximizes the two terms in Proposition 2 of [1], and the defect vanishes at that diagonal. Lemma 1 then gives the full two-unitary optimum. Appending extra zero amplitudes generally changes the multiplicity hypothesis, so this argument applies to the stated ambient dimensions.

For example, on two four-qubit registers, raw amplitudes $(2, \ldots, 2, 1, 0)$ with fourteen twos give a rank-15 pure state. Its optimal purity is 2576747/3518667, which a direct Pauli sum reproduces. Its last dyadic shell has unequal amplitudes, and its rank exceeds the rank-six theorem in [1]. Two exceptions above the background already give a sorted dyadic staircase; the theorem includes previously covered cases.

Scope and replay. General spectra in larger registers and unrestricted rank eight remain open. The argument relies on the cited preprint's analytic reduction and frame bounds. The finite certificate and occupation identities have executable checks. Separate project agents reviewed the physics reduction, tried adversarial complex frames, and replayed the finite checks. External review and publication priority remain open. Run `python3 checks/schmidt7/verify.py` on the research branch. For the kernel replay run `python3 checks/schmidt7/lean_replay.py` with the pinned Lean toolchain. Certificate and checkers.

AI contributions and verification. GPT-6.2 Astra (high) helped draft the discussion of register size. A second project AI reviewer challenged the rank-seven reduction and replayed the coefficient certificate. The pair-sum equality for general rank-seven spectra uses the single unused direction in dimension eight. The larger-register certificates have additional spectral hypotheses, and the two-exception theorem counts zeros in the full ambient spectrum. Rank-seven review and Two-exception review.

Reference. [1] P. Sierant, *Spectral geometry of nonlocal stabilizer entropy*, arXiv:2609.13106v2 (2026). The public problem listing is Open Quantum Problems E27. [2] Liu and Cui, arXiv:2607.10714. [3] Torre, Franchini and Giampaolo, arXiv:2607.07808.