Which Husimi entropy orders can bound Wigner negativity?

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Abstract

Can the spread of heterodyne outcomes bound the Wigner negativity of light? For one optical mode we determine two necessary conditions on the order a of the vacuum-subtracted Husimi Rényi entropy. One photon violates the bound exactly when $0 < a < a_*$, where $a_* = 0.288703\ldots$ is the root of an explicit gamma-function equation. For each fixed $a > 1/2$, including the Shannon and min-entropy limits, a weak coherent superposition violates the bound even as its mean photon number tends to zero. This proves a prediction of Gagatsos. Throughout the surviving interval $[a_*, 1/2]$, we prove the inequality for every coherent or mixed state in the zero–one-photon subspace and for all photon-number-diagonal mixtures through 144 photons. The unrestricted order-one-half conjecture remains open.

A heterodyne detector samples a positive phase-space distribution, the Husimi density. The Wigner density of the same state can have negative regions. Its integral is one, so its absolute integral exceeds one by twice the mass of those regions. The physical question is whether the spread of the measured distribution puts a ceiling on this negative mass. The entropy order selects how much weight to give rare outcomes.

Use the complex coherent amplitude z as phase-space coordinate, with normalized densities $q(z) = \langle z|\rho|z\rangle/\pi$ and W in d^2z. Put $h_a(q) = (1-a)^{-1}\log\int q^a$ and $\Delta_a = h_a(q) - h_a(q_0)$, with the usual Shannon and min-entropy definitions at 1 and ∞. Entropy differences eliminate the constant coordinate Jacobian. Gagatsos asks at which orders $\log\|W\|_1 \le \Delta_a$ holds for every state [1]. Equation (19) and Sec. VI of [1] already exhibit violations at orders $1, 2, \infty$ and identify dilute-cat interference as the mechanism. Section VII predicts failure at every fixed order above one half as the separation grows. Theorem 2 proves that prediction with a single family whose energy also tends to zero.

Theorem 1. *For the single photon the bound fails precisely for $0 < a < a_*$, where a_* is the unique root in $(0,1)$ of*

$$\log\Gamma(1+a) - a\log a = (1-a)\log(4e^{-1/2} - 1).$$

In particular $0.28870 < a_* < 0.28871$.

For number state n, integrating $q_n = \pi^{-1}e^{-|z|^2}|z|^{2n}/n!$ gives

$$\Delta_a(n) = \frac{\log\Gamma(1+an) - a\log(n!) - an\log a}{1-a}.$$

For $n \ge 1$ its numerator $f(a)$ has $f(1) = 0$ and $f''(a) = n^2\sum_{j\ge1}(na+j)^{-2} - n/a < 0$, by comparing the sum with the integral from zero to infinity. Thus the negative secant slope $f(a)/(1-a)$ increases strictly, with its continuous extension at $a = 1$ and limit at infinity. For $n = 1$ it tends to zero at order zero, and at order $1/2$ it equals $\log(\pi/2)$. For a single photon put $u = 2|z|^2$. Its Wigner density times the radial measure is $(2u-1)e^{-u}\,du$. The negative mass is $\int_0^{1/2}(1-2u)e^{-u}\,du = 2e^{-1/2} - 1$. Hence $\|W_1\|_1 = 4e^{-1/2} - 1$ [2], strictly between 1 and $\pi/2$. This proves the root and the sign characterization. The accompanying certificate encloses the two endpoint signs using the Stirling remainder $-1/(360x^3) < \log\Gamma(x+1) - (x+1/2)\log x + x - \frac{1}{2}\log(2\pi) - 1/(12x) < 0$, shifting both gamma arguments by an integer so the constant cancels [3].

Theorem 2. *For every fixed $a \in (1/2, \infty]$, the bound fails for pure states arbitrarily close in trace distance to vacuum with arbitrarily small mean photon number. The same sequence works eventually for each fixed order.*

For $L > 1$ real, let $t = L^{-2}$ and $|\psi_L\rangle = (|0\rangle + t|L\rangle)/\sqrt{N}$, where $N = 1 + t^2 + 2te^{-L^2/2}$. The trace distance to vacuum is $t\sqrt{1-e^{-L^2}}/\sqrt{N}$, and the mean photon number is t^2L^2/N.

Interference contributes at first order. Write $w_u(z) = 2\pi^{-1}e^{-2|z-u|^2}$. The Wigner density multiplied by N is

$$w_0 + t^2 w_L + 2t w_{L/2} \cos(2L \operatorname{Im} z).$$

On the strip $|\operatorname{Re} z - L/2| \le L/4$, the two endpoint Gaussians have exponentially small mass. Outside it the midpoint Gaussian has exponentially small mass. Using $||u+v| - |u| - |v|| \le 2\min(|u|, |v|)$ on each region gives

$$N\|W\|_1 = 1 + t^2 + 2t \int w_{L/2} |\cos(2L \operatorname{Im} z)| \, d^2 z + O(e^{-cL^2})$$

for an absolute $c > 0$. The integral tends to $2/\pi$ by the absolutely convergent Fourier series of $|\cos|$ and the Gaussian characteristic function. Hence $\log \|W\|_1 = (4/\pi)t + o(t)$.

Husimi overlap is small at every fixed order above one half. Write $q = |A + tB|^2/N$ for coherent amplitudes of modulus $|A| = \pi^{-1/2}e^{-|z|^2/2}$ and $|B| = \pi^{-1/2}e^{-|z-L|^2/2}$. For $p = 2a > 1$,

$$\big||u+v|^p - |u|^p - |v|^p\big| \le C_p(|u|^{p-1}|v| + |u||v|^{p-1}).$$

This follows by the mean value theorem, taking the larger modulus first. Both mixed Gaussian integrals decay like $\exp[-(p-1)L^2/(2p)]$ by completing the square. For each fixed finite $a \neq 1$ this gives

$$\Delta_a = \frac{\log(1+t^{2a}) - a\log(1+t^2)}{1-a} + O_a(e^{-c_a L^2}) = o(t).$$

Here $c_a > 0$ may depend on the order. The expansion does not require uniformity as $a \downarrow 1/2$.

At Shannon order, compare to $g = (|A|^2 + t^2|B|^2)/(1+t^2)$. Relative entropy positivity gives $h(q) \le -\int q \log g$. Since $g \ge |A|^2/(1+t^2)$ and $g \le 1/\pi$, $|\log g| \le \log(\pi(1+t^2)) + |z|^2$. The difference $|q - g|$ is bounded by $2t|AB|/N + 2te^{-L^2/2}g/N$. Integrating against the polynomial bound gives $h(q) \le h(g) + O(e^{-cL^2})$ for some $c > 0$. The entropy of the incoherent mixture is at most $h(q_0)$ plus the binary entropy of $t^2/(1+t^2)$, by introducing the classical mixture label. Thus $\Delta_1 \le O(t^2|\log t|) = o(t)$ as an upper bound. At order infinity, evaluating q at $z = 0$ gives $\Delta_\infty \le \log N = O(t^2)$. In each case the positive first-order Wigner term proves strict violation for all sufficiently large L.

Two classes with positive results below order one half

For $0 < a \le 1/2$, $\exp \Delta_a$ is concave in the density matrix. Indeed $\|q\|_a = (\int q^a)^{1/a}$ is concave on nonnegative functions by the reverse triangle inequality (normalize each function to unit quasi-norm and use concavity of x^a). Raising it to the positive power $a/(1-a) \le 1$ preserves concavity; the vacuum factor is constant. Thus positive bounds on pure states extend to their mixtures.

An exact order threshold at a photon cutoff. Among photon-number-diagonal states supported on $0, \ldots, 144$, the tested inequality holds universally for $a \in [a_*, 1/2]$ and there is a counterexample for $0 < a < a_*$ (a single photon witnesses the failure). For every $2 \le n \le 144$ the accompanying interval table encloses a crossing strictly below 0.28870. Strict monotonicity of $\Delta_a(n)$, the preceding concavity, and Theorem 1 give the assertion. Here is how the finite certificate works. In the variable $u = 4|z|^2$, the norm is $\frac{1}{2}\int e^{-u/2}|L_n(u)|du$. Rational sign changes isolate all n Laguerre roots; the degree proves completeness. For $T = L_n + 2T'$ (backward solution on polynomial coefficients), $G = e^{-u/2}T$ has derivative $-e^{-u/2}L_n/2$. Sum its absolute changes over the root intervals, including 0 and ∞. The displaced-parity bound $|e^{-u/2}L_n| \le 1$ controls the root-location error. The two endpoint signs of each entropy comparison use the same Stirling bounds as Theorem 1.

The same sharp threshold on the full single-rail qubit. For states supported on zero and one photon, now allowing arbitrary phase coherence, the inequality holds universally for $a \in [a_*, 1/2]$. At each $0 < a < a_*$ one photon violates it. Thus the single photon determines the exact lower threshold on the whole Bloch ball. We give an analytic reduction and the reproducible finite certificate.

Up to phase rotation the nonvacuum pure states have the form $(b|0\rangle + |1\rangle)/\sqrt{1+s}$, $s = b^2$. Write $d = \mathbb{E}(1-2S)_+$, where S is a mixture of gamma variables of unit scale and shape $K+1$, $K \sim$ Poisson$(s/2)$. The single-rail Wigner formula gives $\|W\|_1 = 1 + 2d/(1+s)$ by completing the square in the Wigner polynomial. Conditional on $K = k$, dropping $e^{-u} \le 1$ in the integral for d_k gives $d_k \le 1/[2^{k+1}(k+2)!] \le 6^{-k}/4$. Thus $d \le e^{-5s/12}/4$. The Husimi integral, by expanding the noncentral complex Gaussian radial density, gives

$$(1-a)\Delta_a = \log\Gamma(1+a) - a\log a + \log F_a(as) - a\log(1+s), \quad F_a(v) = e^{-v}\sum_{k\ge 0}\frac{(1+a)_k v^k}{(k!)^2}.$$

Arbitrarily close to vacuum. For $7/25 \le a \le 1/2$ and $s \ge 9$, the desired inequality holds strictly. To see this, write $e^{(1-a)\Delta_a}$ as the expectation of $|1 + \sqrt{X/(as)}e^{i\theta}|^{2a}$ divided by $(1+1/s)^a$. Here X is unit-mean exponential and θ uniform and independent. For $0 \le y < 1$ the angular mean at radius $\sqrt{y}$ equals $\sum_{j\ge 0}\binom{a}{j}^2 y^j \ge 1 + a^2 y$. For $y \ge 1$ factor out y^a to bound it below by $y^a + a^2 y^{a-1}$. On $[1,4]$ this exceeds or equals $1 + a^2 y$: the difference is concave there since its second derivative has the sign of $-y + a(2-a)$. It vanishes at one and is nonnegative at four because $4^a \ge 1 + 4a^2$ on $[0, 1/2]$ (subtract and use concavity and the two endpoint values). At $y > 4$ the mean is at least one. Integrating yields a numerator at least

$$1 + a/s - (a/s)(1+4as)e^{-4as}.$$

Taylor's theorem gives $1 + a/s - (1+1/s)^a \ge a(1-a)(1+1/s)^{-2}/(2s^2)$. Since $(1+x)^{1-a} \le 1 + (1-a)x$ and $(1+1/s)^a \le 1 + 1/s$, it suffices that

$$\tfrac{a}{2}(1+1/s)^{-2} > \frac{as}{1-a}(1+4as)e^{-4as} + \tfrac{s}{2}e^{-5s/12}.$$

The left side is at least $(7/50)(9/10)^2$. The right side is below $2(2.52)(11.08)/20000 + 9/84$: $v(1+4v)e^{-4v}$ decreases for $v \ge 2.52$, $se^{-5s/12}$ decreases for $s \ge 9$, and the first sixteen exponential-series terms give $e^{10.08} > 20000$, $e^{3.75} > 42$. These bounds separate strictly.

At the one-photon equality point. For $0 \le s \le 1/32$ and $0.28 \le a \le 0.30$, the residual $\Delta_a - \log\|W\|_1$ increases strictly in s. Indeed $F_a(v)$ is the Poisson mean of $g_k = (1+a)_k/k!$. Its derivative is the mean of $g_{k+1} - g_k = ag_k/(k+1)$, at least ae^{-v}, and $F_a(v) \le e^{av}$ by $g_k \le (1+a)^k$. Thus

$$\partial_s \Delta_a \ge [a^2 e^{-as(1+a)} - a]/(1-a) > -8/25,$$

using $e^{-as(1+a)} > 0.98$. In the gamma integrals $d_{k+1} \le d_k/2$ because the extra factor is $u/(k+1)$ on $0 \le u \le 1/2$. Poisson differentiation yields $d' \le -d/4$. Also $d \le 1/4$ and $d \ge e^{-s/2}d_0 > 0.206$, using $d_0 = 2e^{-1/2} - 1 > 0.21$. It follows that

$$-\partial_s \log\|W\|_1 = \frac{2d - 2(1+s)d'}{(1+s)(1+s+2d)} \ge \frac{2(0.206)(5/4)}{(33/32)(49/32)} > 8/25.$$

At $s = 0$ Theorem 1 decides the sign exactly, including equality at a_*.

Compact-region certificate. The remaining rectangles are $[0.29, 0.5] \times [0, 9]$ and $[0.28870, 0.29] \times [1/32, 9]$, in coordinates (a, s). For fixed order, evaluate monotone factors at opposite ends of each s interval. F_a increases because g_k increases; d decreases by the preceding Poisson comparison. Bound the positive series for F_a by partial sums and a geometric tail. Bound d_k by integrating the degree-11 and degree-12 Taylor bounds for e^{-u} against $(1-2u)u^k/k!$ on $[0, 1/2]$, then bound the Poisson tail using

$d_k \le 6^{-k}/4$. The gamma bounds are the same as in Theorem 1. For $a \in [u, v]$ monotonicity of the raw Rényi entropy gives

$$\Delta_a \ge \Delta_v + h_v(q_0) - h_u(q_0), \qquad h_u(q_0) = \log \pi + \frac{\log u}{u - 1}.$$

Monotonicity follows from convexity of $a \mapsto \log \int q^a$ and normalization at $a = 1$. Dyadically subdividing each rectangle until the lower bound exceeds $\log \|W\|_1$ exhausts both rectangles. The certificate is archived as `sharp-domain.json`; `checks/sharp_qubit.py` rebuilds every leaf from its root using rational arithmetic with outward-rounded logs and exponentials. The two analytic regions close the domain. Concavity of e^{Δ_a} and convexity of the Wigner norm extend the result to mixed states. The unrestricted optical conjecture still permits other witnesses.

What remains open. At $a = 1/2$ a separated component of amplitude t contributes to the integral of $\sqrt{q}$ at order t. This is the same order as the Wigner interference term. The argument of Theorem 2 therefore stops precisely where Gagatsos's conjecture begins. A counterexample anywhere in $[a_*, 1/2]$ must leave the zero–one-photon subspace and the tested class of diagonal mixtures.

Computational record. The two compact covers contain 18 strips and 10,467 certified leaves. In a checkout of the archived verifier revision, with Python and SciPy installed, run

```
python checks/sharp_qubit.py
python renyi_scan.py
```

The first command reconstructs the compact covers and compares the counts strip by strip. The second reconstructs all 144 number-state brackets and checks their ordering. SciPy proposes roots. Rational sign tests and outward enclosures certify the comparisons. The asymptotic and boundary arguments are given in the text.

Contributions. Aryan Singh initiated and directed the oo research project. GPT-6.2 Astra (high) was used for mathematical reasoning and manuscript preparation. The computer-assisted portions of the proof are identified above.

References

[1] C. N. Gagatsos, *Wigner negativity costs phase-space area*, Secs. VI–VII, especially open question (iv) (2026). arXiv:2611.30969v1.

[2] A. Kenfack and K. Życzkowski, J. Opt. B **6**, 396 (2004). arXiv:quant-ph/0406015.

[3] NIST Digital Library of Mathematical Functions, §5.11, Stirling expansion with remainder on the positive real axis.