A sharp Wigner–Husimi bound for phase-invariant light

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Abstract

Gagatsos conjectured an upper bound on single-mode Wigner negativity in terms of the Husimi distribution. We prove the conjectured inequality for every photon-number-diagonal state, with no cutoff on photon number. The vacuum is the unique state in this class attaining finite equality. The proof bounds a Laguerre absolute integral by an exactly computable fourth moment and compares that moment with the Husimi integral by induction.

1 What can a heterodyne distribution tell us?

Let W_ρ be the Wigner function and Q_ρ the probability density of an ideal heterodyne measurement on one optical mode. We use $[x,p]=i$, $z=(x,p)$, $\alpha=(x+ip)/\sqrt{2}$ and $Q_\rho(z)=\langle\alpha|\rho|\alpha\rangle/(2\pi)$. Both distributions have integral one in $d^2z=dx\,dp$. If W_ρ is integrable, its negative volume is $(\|W_\rho\|_1-1)/2$. Define

$$H(\rho)=\frac{1}{8\pi}\left(\int_{\mathbb{R}^2}\sqrt{Q_\rho(z)}\,d^2z\right)^2.$$

Gagatsos conjectured $\|W_\rho\|_1\le H(\rho)$ for every single-mode state and proved $\|W_\rho\|_1\le 2H(\rho)$ in general [1]. States with nonnegative Wigner functions already obey the conjecture by the Lieb–Solovej bound cited there. Excited photon-number states have negative Wigner regions [2].

Theorem 1. *For every state invariant under all optical phase rotations, $\|W_\rho\|_1\le H(\rho)$ in the extended nonnegative reals. When $H(\rho)<\infty$, equality holds exactly for the vacuum. The constant $1/(8\pi)$ is sharp.*

Phase rotation acts on $|n\rangle$ by $e^{-in\theta}$. Invariance for every θ forces every off-diagonal density-matrix element to vanish. Thus the class consists precisely of arbitrary probability mixtures of photon-number states.

2 An all-orders estimate for a number state

Write $r=|z|$, and let L_n be the ordinary Laguerre polynomial with $L_n(0)=1$. The number-state densities are

$$W_n(z)=\frac{(-1)^n}{\pi}e^{-r^2}L_n(2r^2),\qquad Q_n(z)=\frac{e^{-r^2/2}}{2\pi n!}\left(\frac{r^2}{2}\right)^n.$$

In polar coordinates substitute $u=2r^2$ in the Wigner integral and $v=r^2/2$ in the Husimi integral. The resulting quantities are

$$A_n=\frac{1}{2}\int_0^\infty e^{-u/2}|L_n(u)|\,du,\qquad B_n=H(|n\rangle\langle n|)=\frac{2^n\Gamma(1+n/2)^2}{n!}. \tag{1}$$

In particular $A_0=B_0=1$. For $f_n(u)=e^{-u/2}L_n(u)$ the Laguerre orthogonality relation gives $\int f_n^2=1$. Let $M_n=\int_0^\infty u^4 f_n(u)^2\,du$. For $a>0$, Cauchy–Schwarz with weight $1+(u/a)^4$ gives

$$4A_n^2\le\left(1+\frac{M_n}{a^4}\right)\int_0^\infty\frac{du}{1+(u/a)^4}=\left(1+\frac{M_n}{a^4}\right)\frac{\pi a}{2\sqrt{2}}.$$

Here the last integral follows from the beta integral. Taking $a = (3M_n)^{1/4}$ and raising to the fourth power yields

$$A_n^8 \le \frac{\pi^4}{1728} M_n. \tag{2}$$

To evaluate M_n, apply the three-term recurrence [3]

$$uL_j = -(j+1)L_{j+1} + (2j+1)L_j - jL_{j-1}$$

twice and take the squared norm of $u^2 L_n$ in $L^2(e^{-u}du)$. In order of Laguerre indices $n-2, \ldots, n+2$, the coefficients are

$$n(n-1), \quad -4n^2, \quad 6n^2+6n+2, \quad -4(n+1)^2, \quad (n+1)(n+2).$$

Negative indices are omitted and their coefficients vanish. Summing squares gives

$$M_n = 70n^4 + 140n^3 + 170n^2 + 100n + 24.$$

The gamma recurrence in (1) gives $B_{n+2}/B_n = (n+2)/(n+1)$. Consequently $B_{n+2}^8/M_{n+2} > B_n^8/M_n$ whenever

$$D(n) = (n+2)^8 M_n - (n+1)^8 M_{n+2} > 0.$$

The coefficients of D in increasing powers of n are

$$(3000, 20324, 60606, 102244, 107730, 74088, 33516, 9672, 1620, 120).$$

Every coefficient is positive. For the odd base, $M_1 = 504$, $B_1 = \pi/2$ and $1728 \cdot 3^4 > 2^8 \cdot 504$; hence $B_1^8 > \pi^4 M_1/1728$. For the even base, $M_2 = 3144$, $B_2 = 2$ and $1728 \cdot 2^8 > 100 \cdot 3144$. Here $3 < \pi < 22/7$ implies $\pi^4 < 100$. Induction on each parity and (2) give $A_n < B_n$ for every $n \ge 1$.

3 Arbitrary photon-number statistics

Let $\rho = \sum_{n\ge 0} p_n |n\rangle\langle n|$. Both phase-space functions are linear in ρ; pointwise convergence follows from trace-norm convergence and the bounded displaced parity and coherent projectors defining them. The triangle inequality and Tonelli's theorem give $\|W_\rho\|_1 \le \sum_n p_n A_n$. To bound the other side, apply the integral triangle inequality for the Euclidean norm to the vector with components $\sqrt{p_n Q_n(z)}$, $0 \le n \le N$. This gives

$$\left(\sum_{n=0}^{N} p_n \left[\int \sqrt{Q_n} \right]^2 \right)^{1/2} \le \int \sqrt{\sum_{n=0}^{N} p_n Q_n} \le \int \sqrt{Q_\rho}.$$

Taking $N \to \infty$ and squaring proves

$$\|W_\rho\|_1 \le \sum_n p_n A_n \le \sum_n p_n B_n \le H(\rho).$$

If $H(\rho)$ is finite and some $p_n > 0$ for $n \ge 1$, the middle inequality is strict. The vacuum saturates it as well as both endpoint bounds. This proves the theorem.

Physical check. For one photon, $L_1(u) = 1-u$. Splitting (1) at $u = 1$ gives $A_1 = 4e^{-1/2} - 1$, whereas $B_1 = \pi/2$. The Wigner negative volume is $2e^{-1/2} - 1 \approx 0.2131$ and the heterodyne upper bound is $(\pi/2 - 1)/2 \approx 0.2854$. This one-photon value is known [2]. The theorem establishes the bound for every photon count and every probability distribution over counts. It leaves the general question for photon-number coherences open. Applying the same displacement to the whole state simply translates both integrals.

AI contributions and verification. GPT-6.2 Astra (high) helped draft the explanation of how the number-state bounds extend to countable mixtures. Lean checks the arithmetic induction step and its base values. The phase-space identities and the integral triangle inequality that passes to a diagonal mixture are written arguments. Photon-number coherences require a separate analysis. Verification record.

References

[1] C. N. Gagatsos, *Wigner negativity costs phase-space area*, Conjecture 1, Eq. (7), and Propositions 1–2 (2026). arXiv:2611.30969v1.

[2] A. Kenfack and K. Życzkowski, *Negativity of the Wigner function as an indicator of nonclassicality*, J. Opt. B **6**, 396–404 (2004). arXiv:quant-ph/0406015.

[3] NIST Digital Library of Mathematical Functions, §18.3 and §18.9, Laguerre orthogonality and recurrence.