Exact finite maxima for positive coordinate supports at the Werner endpoint

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Abstract

For k copies of a d-dimensional Werner state at the one-copy distillability boundary, we compute the largest computational-basis support on Alice's side for which the full compression is positive semidefinite. For three copies and every $d \geq 2$ we derive the exact maximum $\max(8, 2d)$. The finite table supplies witnesses and replayable upper-bound certificates. The object being maximized is the cardinality of a set of basis words. This finite calculation leaves distillability by Schmidt-rank-two vectors open.

Certified values

Local dimension d	Copies k	Maximum support size
5	3	10
6	3	12
7	3	14
8	3	16
4	4	16

The elementary constructions are useful controls. A binary box gives 2^k words. Repeating a symbol on two sites and using a binary box on the remaining sites gives $d\,2^{k-2}$ for $k \geq 2$. In three copies this second construction has $2d$ words. For the three-copy rows the matching argument below proves optimality in every dimension. The four-symbol rows also follow from the companion paper. The enumerations independently check these results.

Reduction and certificates

Use the unnormalized $\Omega = \sum_i |ii\rangle$ and write $K = I - |\Omega\rangle\langle\Omega|/2$. The companion manuscript proves that compression of $K^{\otimes k}$ to rows indexed by S, with Bob unrestricted, is positive semidefinite exactly when each triple of words in S has an even number of coordinates with three distinct symbols. By independent symbol relabeling in each coordinate one word may be fixed at zero. Two other words are compatible when their triangle with zero has even parity. An anchored clique meets the criterion for all triples: the product of three anchored triangle signs gives the sign of the triangle between their non-anchor vertices.

The search branches on a vertex, considering all cliques which contain it and all which exclude it. At each terminal branch the remaining candidates are partitioned into independent sets. Each clique takes at most one vertex from each set. Adding the included vertices gives the certified upper bound. An explicit clique witnesses the lower bound. Some instances are divided into layers by the maximum Hamming weight t relative to the anchor. A word of weight t can be mapped to t leading ones and trailing zeros while fixing the anchor. Exhausting $t = 1, \ldots, k$ covers every nonsingleton support. A reported maximum requires all layers.

A uniform check for three copies

For every $d \geq 2$ the three-copy maximum is $\max(8, 2d)$. The binary box attains eight. Words (i, i, z) with z in a fixed two-symbol alphabet attain $2d$. Here is the upper bound directly from the companion paper's even-triple criterion.

If a coordinate uses at most two symbols, projecting it away preserves admissibility: lift any three distinct projected words and use that the removed coordinate cannot contribute an all-distinct triple. Each projected word has at most two lifts. The two-copy bound $\max(4, d)$ from the companion paper gives the result.

Otherwise every coordinate uses at least three symbols. No two words can differ at just one position: choosing a third word with a third symbol at that position would give exactly one all-distinct position. For each fixed first symbol the other two coordinates therefore form a matching, viewed as edges between two alphabets. If every matching has size at most two, there are at most $2d$ words. Fix a larger matching of size $t \geq 3$, with edges (b_i, c_i). For any word outside its fiber, write its edge as (u, v). The triple with matching edges i, j gives

$$u \in \{b_i, b_j\} \quad \Longleftrightarrow \quad v \in \{c_i, c_j\}.$$

Membership in all two-subsets of $[t]$ distinguishes each index and distinguishes it from an absent endpoint. Hence an outside edge using a matched endpoint would repeat the entire edge, giving two words that differ only at the first position. All outside edges avoid both sets of matched endpoints.

Two edges from different outside fibers must now share exactly one endpoint: test their words against a word in the large fiber. The first symbols are all distinct, so exactly one of the other two positions must also be all distinct. If an outside fiber has three or more edges, an edge in a further fiber could meet at most two of that matching's edges. Thus there are at most two nonempty fibers overall and at most $2d$ words. If the largest outside matching has size two, its two edges can be met simultaneously only by the two cross edges. Neither cross edge can recur in different fibers. This gives at most $t+4 \leq \max(8, 2d)$ words. If all outside fibers have size at most one, the bound is $t+d-1 \leq 2d$. This exhausts the cases.

Reproducibility and scope

The auditor rebuilds compatibility from the parity of three Hamming distances, replays each decision with sets, validates every color class, checks prefix coverage, and reconstructs the lower witness. An independent matrix test also checks the physical reduction on every two-copy qutrit support. Run `python3 continuous.py audit` from the repository root. The accompanying certificates specify every search leaf. The certificates prove finite maxima for this specified basis and endpoint. Priority in the combinatorics or quantum information literature remains unassessed. This is an internally reviewed working paper.

AI contributions and verification. GPT-6.2 Astra (high) helped draft the explanation of the support certificates. For each table entry, the auditor checks the deposited lower witness, verifies each color class and confirms that the branches cover the search. The separate matching argument gives $\max(8, 2d)$ at three copies, including dimensions outside the table. These checks concern sets of basis words. Table and audit procedure and Branch-certificate checker.

Related formulation: P. Costa Rico, *New partial trace inequalities and distillability of Werner states* (2025), doi:10.1007/s11005-025-01935-y. See also the companion manuscript, *Positive coordinate compressions of tensor powers of a partially transposed Werner state*, in this repository.