Optimal stabilizer check weights at distance four: the [[12,4,4]], [[12,3,4]] and [[10,2,4]] cases

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Abstract

Twelve data qubits can encode four logical qubits at distance four using checks of weight at most seven. We exhibit a stabilizer code attaining seven, improving the upper bound eight in Table 2 of Wei et al. With three logical qubits the optimum is six, and with one logical qubit at lengths ten through twelve it is four. A graph-state census shows that every $[[10, 2, d \geq 4]]$ stabilizer has the same weight enumerator and requires generators of weight at least six. The twelve-qubit bounds follow from explicit Pauli generators and rational weight-enumerator certificates.

Three attaining codes

The weight of a Pauli check counts its nonidentity positions. Let $W_{\mathrm{opt}}(n, k, d)$ minimize the largest generator weight over all generating bases of all binary stabilizer codes with parameters $[[n, k, d]]$. The three columns below are separate generating sets. Give each Hermitian Pauli string the positive sign. The number on its right is its weight.

	$[[12,4,4]]$		$[[12,3,4]]$		$[[10,1,4]]$	
1	`IIIIIIYYZZXX`	6	`ZIIYIYIXYIIZ`	6	`YIIYYIXIII`	4
2	`ZZZZZZIIIIII`	6	`IIYIYXIIXYXI`	6	`IIIYIIIXXX`	4
3	`ZXXXIXXIXIII`	7	`ZIIZIXIZIIXY`	6	`IIZIIXIZIZ`	4
4	`XIXIYXIZZIIY`	7	`XXYIZIIYYIII`	6	`IIXIYYIIXI`	4
5	`IIYXXXZZIIXI`	7	`XIIXIZZYIXII`	6	`IYIZXIIIZI`	4
6	`XXIZXXIIIIYZ`	7	`IYYIIZIZIYYI`	6	`IXYIIIIYYI`	4
7	`IYIIIXZYXIYY`	7	`IIIIZXIZXXIY`	6	`XXIIZXIIII`	4
8	`XZXXXIIXIXII`	7	`XZZIIXYIIIZI`	6	`ZIYIIZYIII`	4
9			`IYZZZXZIIIII`	6	`IXIXIIYIIZ`	4

The rows within each column commute. Their products give 256, 512 and 512 distinct strings modulo phase, respectively. Independence implies a simultaneous $+1$ eigenspace of dimension 2^k. To check distance, enumerate every support of size $i = 1, 2, 3, 4$ and every assignment of X, Y, Z to it. Count errors that commute with every row and lie outside its unsigned product group. Each column gives zero through weight three. At weight four the counts are 225, 105 and 40. Examples are `YYZZIIIIIIII`, `ZYYIIIXIIIII` and `YYXIIIIIIX`, respectively.

How the first code was found. Starting with Grassl's $[[12, 4, 4]]$ group, retain a rank-seven subgroup. Solve linear equations for a replacement check commuting with this subgroup and detecting every weight-one to weight-three error that commutes with it. Enumerate the resulting groups, then take independent generators in increasing order of weight. The replay script recovers the first column.

Twelve-qubit lower bounds by exact identities

Let S be a stabilizer group of size $q = 2^{12-k}$, taken modulo phases, with distance at least four. Let A_j count its elements of weight j and $T_w = \sum_{j=1}^{w} A_j$. To generate S, at least $12 - k$ independent elements

are required. Set $K_{ij} = [z^i](1+3z)^{12-j}(1-z)^j$ and define

$$H_i = \sum_j K_{ij} A_j - q A_i, \qquad C_i = \sum_j (-1)^j K_{ij} A_j.$$

These are unnormalized normalizer-difference and shadow enumerators. For an error E let $s(E,g)$ be its commutator bit against $g \in S$. Summing $(-1)^{s(E,g)}$ over all errors of weight i gives K_{ij} for g of weight j by multiplying one-site polynomials. Summing over g first gives q for each commuting E and zero otherwise. Distance at least four implies $H_0 = H_1 = H_2 = H_3 = 0$.

Weight parity on S is linear, since $\mathrm{wt}(gh) \equiv \mathrm{wt}(g) + \mathrm{wt}(h) + s(g,h) \pmod 2$ and $s(g,h) = 0$ on S. By character orthogonality, the sum of $(-1)^{s(E,g)+\mathrm{wt}(g)}$ over $g \in S$ is zero or q. Summing over errors of weight i gives $C_i \geq 0$. The even-weight count $P = \sum_{j\,\mathrm{even}} A_j$ equals q in the all-even case and $q/2$ in the mixed case.

Four logical qubits ($q = 256$). For all-even groups,

$$512(T_6 + 6A_2 + 3A_4) = 3072A_0 + 52H_0 + 19H_1 + 6H_2 + H_3.$$

Thus $T_6 \leq 6$. In the mixed case set $D = 19A_1 + 6A_2 + 10A_3 + 3A_4 + 2A_5$. Then

$$512(T_6 + D) + C_1 + C_3 = -1024A_0 + 36H_0 + 20H_1 + 6H_2 + 2H_3 + 16P.$$

Here $P = 128$ and $A_0 = 1$, so $T_6 \leq 2$. Both cases exclude the eight independent nonidentity checks required at weight six or less. Hence $W_{\mathrm{opt}}(12,4,4) = 7$.

Three logical qubits ($q = 512$). On even coordinates the following linear-form identity holds:

$$2048T_5 = 16384A_0 + 52H_0 + 19H_1 + 6H_2 + H_3 - 512A_6.$$

Since $A_0 = 1$ and $H_0 = \cdots = H_3 = 0$, this gives $T_5 = 8 - A_6/4 \leq 8$.

In the mixed case define the nonnegative quantity

$$R = 19968A_1 + 16896A_3 + 1024A_4 + 2048A_6 + 11C_1 + 2C_2 + 5C_3.$$

The following identity holds on all thirteen coordinates:

$$7168T_5 + R = 45056A_0 + 168H_0 + 87H_1 + 26H_2 + 9H_3 + 40P.$$

Substituting $P = 256$ gives $T_5 \leq 55296/7168 = 54/7 < 8$. All four identities follow by comparing coefficients of A_j for $0 \leq j \leq 12$ (only even j for all-even groups). The exact-arithmetic checker implements the dual bound. Thus in either case there are too few nonidentity low-weight elements to span a rank-nine group. The displayed certificate attains six, proving $W_{\mathrm{opt}}(12,3,4) = 6$.

Ten qubits and two logical qubits

AI contributions and verification. GPT-6.2 Astra (high) helped draft the comparison of upper and lower bounds. For the two twelve-qubit codes in the title, explicit generators establish attainability and weight-enumerator identities exclude lighter bases. The ten-qubit census was implemented separately in Python and C++. Both calculations depend on the completeness of the Danielsen–Parker graph-state list. Census comparison.

Every binary $[[10, 2, d \geq 4]]$ stabilizer has unsigned weight enumerator

$$A(z) = 1 + 90z^6 + 135z^8 + 30z^{10}.$$

We obtain this statement by a finite calculation conditional on the completeness of the Danielsen–Parker classification of self-dual additive codes under local Clifford operations and qubit permutations. We use the graph-parent and syndrome method employed by Zhang for twelve qubits, here at length ten.

Completeness of the reduction. Extend any rank-eight stabilizer S to a maximal isotropic subspace L of the binary Pauli space. Up to the operations above, L is the stabilizer of a graph state with symmetric zero-diagonal adjacency matrix Γ. Parameterize its elements by $g(u) = (u, \Gamma u)$ for binary ten-vectors u, using ordered X, Z coordinates. Then $S = \{g(u) : u \in U\}$ for an eight-dimensional subspace U, uniquely specified by the two-dimensional binary subspace $C = U^{\perp}$ under the ordinary dot product. The code distance and all unsigned stabilizer weights are invariant under these transformations.

An error $E = (x, z)$ commutes with all of S exactly when $s = z + \Gamma x$ belongs to C. If $s \neq 0$, this error lies outside L and hence outside S. Therefore every nonzero syndrome of an error of weight one through three must avoid C. When $s = 0$, the error equals $g(x)$; distance at least four requires $x \in U$, or equivalently $c \cdot x = 0$ for every $c \in C$. These two conditions are necessary and sufficient and allow short stabilizer elements. Signs do not affect either test.

Extent of the census. The Danielsen database supplies 3,132 connected order-ten graph representatives. Taking unordered multisets of smaller connected representatives with total order ten adds 858 disconnected ones; local complementation and vertex permutation preserve the multiset of connected component orbits. For each of the 3,990 parents, form the two syndrome tests for every one of the 3,675 unsigned errors of weights one to three. Enumerate C by triples $a < b < a + b$ of binary integers whose three nonzero elements all pass the tests (addition is XOR). Exactly 47 planes across ten graph representatives survive. These count presentations; one code can have several maximal parents.

For each surviving plane, enumerate the 256 graph stabilizers indexed by $u \in U$ and count $\mathrm{wt}(g(u))$ by the size of the union of the supports of u and Γu. Every plane gives the polynomial above. Thus all nonidentity stabilizer elements have weight at least six. Grassl's archived group has a weight-six basis, which proves $W_{\mathrm{opt}}(10, 2, 4) = 6$. The published census package checks SHA-256 for all ten source files and for the assembled parent list. The check relies on the published parent classification, independently of the paper's two rational bounds.

Optimality and source comparison

Wei et al. report $[7, 8]$ for $W_{\mathrm{opt}}(12, 4, 4)$, $[4, 6]$ for $W_{\mathrm{opt}}(n, 1, 4)$ at $n = 10, 11, 12$, and $[6, 8]$ for $W_{\mathrm{opt}}(12, 3, 4)$. Their Theorem 12 shows that any nontrivial stabilizer code generated at weight at most three has distance at most two. This analytic result supplies the lower bound four in all three one-logical-qubit cases.

Append one qubit in $|0\rangle$ to the ten-qubit certificate. Extend all old checks by identity and add Z on the new qubit. A Pauli commuting with the added Z acts by I or Z there. Multiplying by the new check therefore removes its action on that qubit, preserving its logical class and weakly decreasing its weight. Every old logical has a representative of the same weight on the enlarged code. The distance stays four, and the maximum check weight stays four. Iterating gives

$$W_{\mathrm{opt}}(10, 1, 4) = W_{\mathrm{opt}}(11, 1, 4) = W_{\mathrm{opt}}(12, 1, 4) = 4.$$

For the two twelve-qubit cells, Wei et al. report the lower bounds seven and six, respectively, using their weight-enumerator linear-programming procedure (Appendix L, Algorithm 1). The rational identities above give direct checks of both bounds using the enumerator conditions of their Appendix G. The constructions attain both lower bounds.

Attribution of the subcode construction. Wei et al., Appendix J, prescribe appending a weight-four logical Pauli to the very same $[[12, 4, 4]]$ parent to obtain their $[[12, 3, 4]]$ code. We minimize the maximum generator weight across choices of logical Pauli and across generating bases. Of the parent's 75 nonzero logical cosets with a weight-four representative, 60 give maximum weight six after basis minimization and 15 give eight. In the site order obtained by reversing Grassl's printed matrix, appending `XXIIIIZZIIII` gives one of the 60. Likewise appending `IZXIZZZIZI` to Grassl's $[[10, 2, 4]]$

group yields a $[[10, 1, 4]]$ code with maximum weight four. These are alternatives to the second and third columns on page one.

The first column instead preserves four encoded qubits by changing one stabilizer constraint of the parent. The parent admits an all-even stabilizer group, whereas the first column contains odd-weight elements. Qubit permutations and local Clifford gates preserve weight, so these two groups are inequivalent under those operations. Jing and Tu updated the distance-three entries of the source table and identified distance four as further work. Our comparison establishes improvements to the cited bounds. Historical priority for the values and code classes remains undetermined.

All sixteen intervals in the source table

The table covers each entry of Wei et al., version 2, Table 2 that lists two distinct finite endpoints. The scope is $10 \leq n \leq 12$ within that table. An interval changes either because a code attains a previous lower bound or because a proof excludes smaller weights. Repeated numerical optima can follow from a single construction.

n	k	d	Source interval	Optimum	Evidence
10	4	3	6–7	6	Jing–Tu
11	3	3	4–5	5	Zhang; Jing–Tu
11	4	3	5–7	5	Jing–Tu
11	5	3	6–8	7	Jing–Tu
12	3	3	4–5	4	Jing–Tu
12	4	3	5–7	5	Jing–Tu
12	5	3	6–8	6	Jing–Tu
12	6	3	8–10	8	Jing–Tu
10	1	4	4–6	4	Construction and Wei et al., Theorem 12
10	2	4	5–6	6	Census and Grassl construction
11	1	4	4–6	4	Padding the ten-qubit construction
11	2	4	4–6	5	Zhang
12	1	4	4–6	4	Padding the ten-qubit construction
12	2	4	4–6	5	Zhang
12	3	4	6–8	6	Construction and rational bound
12	4	4	7–8	7	Construction and rational bound

Sources

F. Wei, Z. Han, A. Y. He, Z. Li and Z.-W. Liu, *Theory of low-weight quantum codes*, v2 (2026), Table 2, Theorem 12 and Appendices G,J,L.

Y. Jing and Y. Tu, proof manuscript and updated distance-three table and QIQCOP submission 111.

Y. Zhang, *Optimal check weight for two-logical-qubit codes of lengths eleven and twelve* (17 November 2026), and the $[[11, 3, 3]]$ exclusion.

L. E. Danielsen and M. G. Parker, *On the classification of all self-dual additive codes over GF(4) of length up to 12*, J. Combin. Theory Ser. A **113**, 1351–1367 (2006). Graph-state representatives and data documentation.

M. Grassl, *Code tables*, matrices for $[[10, 2, 4]]$ and $[[12, 4, 4]]$. The QEC Challenge also catalogs these parent parameter sets, length ten and length twelve.