The refined BMV trace inequalities for a matrix with two spectral values

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26 November 2026

Abstract

Let A, B be positive semidefinite matrices of any finite size. If A has at most two distinct eigenvalues, the mean trace of all words with n letters A and m letters B lies between $\mathrm{Tr}[(A^{n/m}B)^m]$ and $\mathrm{Tr}(A^nB^m)$ for all positive integers n, m. This follows by grouping Hägele's path expansion according to the number of visits to one eigenspace. The resulting weights are coefficients of an ordinary Bessis–Moussa–Villani polynomial and are therefore nonnegative. Together with a known three-eigenvalue counterexample, this gives the sharp spectral threshold for the upper bound. An appendix independently reproduces finite cases of a previously established three-eigenvalue lower bound.

The two-sided bound

Write $\mathcal{A}_{n,m}(A,B) = \binom{n+m}{m}^{-1} \sum_W \mathrm{Tr}\, W$, where W runs over all words with n letters A and m letters B. Hägele's refinement of the BMV conjecture asks whether, for Hermitian H, K, the value of $\mathcal{A}_{n,m}(e^H, e^K)$ lies between the quantum partition function $\mathrm{Tr}\, e^{nH+mK}$ and $\mathrm{Tr}(e^{nH}e^{mK})$ [1]. Cha and Lee disproved the upper inequality in general [4]. Here the spectral multiplicities and both word counts are arbitrary.

Theorem. If $A, B \in M_d(\mathbb{C})$ are positive semidefinite and A has at most two distinct eigenvalues, then

$$\mathrm{Tr}[(A^{n/m}B)^m] \;\le\; \mathcal{A}_{n,m}(A,B) \;\le\; \mathrm{Tr}(A^nB^m) \qquad (n,m \ge 1). \tag{1}$$

For $A = e^H$ and $B = e^K$, the first term dominates $\mathrm{Tr}\, e^{nH+mK}$ by the Araki–Lieb–Thirring inequality and the Lie–Trotter formula [2, Lemma 2]. Interchanging the two factors gives the partition-function and upper bounds also when K has at most two spectral values.

Proof. Write $A = aP + bQ$ with P, Q complementary orthogonal projections and $a, b \ge 0$. A scalar A gives equality. Let $\mathcal{R} = \{r \in \mathbb{N}_0^m : \sum_i r_i = n\}$, of size $N = \binom{n+m-1}{m-1}$. By cyclicity [2, Lemma 3],

$$\mathcal{A}_{n,m}(A,B) = N^{-1} \sum_{r\in\mathcal{R}} \mathrm{Tr}(A^{r_1}BA^{r_2}B\cdots A^{r_m}B).$$

For example, marking one of the m occurrences of B in each word counts each word m times. Rotating the marked letter to the end leaves a weak composition and a rotation offset in $\{0,\dots,n+m-1\}$. The trace is independent of that offset.

For a sequence $E = (E_1,\dots,E_m) \in \{P,Q\}^m$, set $w(E) = \mathrm{Tr}(E_1BE_2B\cdots E_mB)$. Let $\lambda_i = a$ when $E_i = P$ and $\lambda_i = b$ when $E_i = Q$. The scalar multiplying $w(E)$ in the average is

$$\kappa_k = N^{-1} \sum_{r\in\mathcal{R}} \prod_i \lambda_i^{r_i}, \qquad k = \#\{i : E_i = Q\}.$$

It depends only on k because $\mathcal{R}$ is invariant under coordinate permutations. With $c_k = \sum_{E:\#Q=k} w(E)$ we therefore have $\mathcal{A}_{n,m}(A,B) = \sum_k c_k\kappa_k$. Terms with exponent zero equal one, including at a zero eigenvalue.

Positivity of the grouped weights. Put $U = B^{1/2}PB^{1/2}$ and $V = B^{1/2}QB^{1/2}$. Both are positive semidefinite, and cyclicity gives

$$p(t) = \operatorname{Tr}[(U+tV)^m] = \operatorname{Tr}[((P+tQ)B)^m] = \sum_{k=0}^{m} c_k t^k.$$

The polynomial form of the ordinary BMV theorem gives $c_k \geq 0$ [3]. This use of the theorem permits arbitrary ranks of P and Q and arbitrary complex off-diagonal blocks of B.

Scalar bounds. Every coordinate r_i has average n/m over $\mathcal{R}$. The arithmetic–geometric mean inequality over compositions gives the lower bound below. For the upper bound, apply the weighted arithmetic–geometric mean inequality to $\prod_i (\lambda_i^n)^{r_i/n}$ for each composition and average:

$$(a^{m-k}b^k)^{n/m} \;\leq\; \kappa_k \;\leq\; \frac{(m-k)a^n + kb^n}{m}.$$

Continuity extends the lower bound from positive eigenvalues to zero. After multiplication by c_k and summation, expansion of $(A^{n/m}B)^m$ gives the left side of (1). For the upper side, $p(1) = \operatorname{Tr} B^m$ and $p'(1) = m \operatorname{Tr}[V(U+V)^{m-1}] = m \operatorname{Tr}(QB^m)$. Therefore

$$\sum_k c_k \frac{(m-k)a^n + kb^n}{m} = a^n \operatorname{Tr}(PB^m) + b^n \operatorname{Tr}(QB^m) = \operatorname{Tr}(A^n B^m). \quad \square$$

An elementary check through five letters B. In the splitting $P \oplus Q$, let B have diagonal blocks X, Y and upper-right block Z. With $m \leq 5$, a cyclic sequence of labels switches blocks zero, two or four times. With zero switches its weight is $\operatorname{Tr} X^m$ or $\operatorname{Tr} Y^m$. Two switches give $\operatorname{Tr}(X^i Z Y^j Z^*)$ for some $i, j \geq 0$. With four switches there is at most one diagonal step, giving $\operatorname{Tr}[(ZZ^*)^2]$, $\operatorname{Tr}[X(ZZ^*)^2]$, or $\operatorname{Tr}[Y(Z^*Z)^2]$, up to cyclic rotation. These traces are nonnegative because X, Y are positive semidefinite and the trace of a product of two positive semidefinite matrices is nonnegative. Thus in these low degrees one obtains the required weight signs directly. If one eigenspace is one-dimensional, one can diagonalize the other diagonal block and adjust basis phases to make all entries nonnegative. The general theorem allows both eigenspaces to have arbitrary rank.

Scope, sharpness and an independent finite reproduction

The theorem is a direct corollary of the composition identity and ordinary BMV coefficient positivity. Priority for this particular deduction has not been established. Since $\mathcal{A}_{n,m}(A,B) = \mathcal{A}_{m,n}(B,A)$, a violation of the upper inequality requires at least three spectral values in *each* factor. Cha and Lee's example [4] attains that threshold: take $A = 1 \oplus xC$, $B = xC \oplus 1$, $C = \left(\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix}\right)$ and $x = 10^{-3}$. Both spectra equal $\{0, 2x, 1\}$. At $n = m = 5$ their example has $\mathcal{A}_{5,5}(A,B) - \operatorname{Tr}(A^5B^5) > 0$ [4, Proposition 6]. Adding εI to both matrices preserves the strict polynomial gap for sufficiently small $\varepsilon > 0$. Both then have three distinct positive eigenvalues and Hermitian logarithms. Thus the threshold also holds for the original exponential inputs. The lower bound for general spectra remains open.

The lower comparison at $m = 4$ is covered by an existing three-value containment theorem. At arbitrary word counts, the lower comparison for $A = tP$ follows from the spectral-pinching inequality, and a one-dimensional eigenspace permits the nonnegative-entry basis of [2, Proposition 1]. The priority audit found no previous general deduction for two positive levels of unrestricted multiplicities and all $m \geq 5$.

AI contributions and verification. GPT-6.2 Astra (high) helped draft the explanation of grouped spectral paths. An internal AI reviewer examined the appendix case with twelve letters A and three letters B. Its exact coefficient test reproduces a previously proved case. The main two-eigenvalue argument uses the ordinary BMV theorem to establish the sign of each group total. Coefficient review.

Appendix. A public Lean proof [5] covers all positive integers n with $m = 3$ when A has at most three distinct eigenvalues, in any finite dimension. The following direct computation reproduces the subfamily $d = 3$, $11 \le n \le 36$ using Hägele's sufficient criterion [2, Lemmas 5–6]. For $a \le b \le c$ the eigenvalues of A, put

$$F(u,v,w) = \sum_{i+j+k=n} u^i v^j w^k - \tbinom{n+2}{2}(uvw)^{n/3}.$$

All exponents in the sum are nonnegative integers. Define

$$\begin{aligned} X &= F(a,a,c), \qquad Y = F(b,b,c), \qquad Z = F(a,b,c), \\ Q_0 &= F(b,b,a)F(c,c,a) + F(a,a,b)F(c,c,b), \\ D &= Z^2 - XY, \qquad T = Z(X+Y) + 2XY, \qquad L = Q_0 T^2 - D^2(X+Y)^2. \end{aligned}$$

Nonnegativity of L suffices for the lower inequality in (1) for $m = 3$ under these appendix hypotheses. Indeed L is Hägele's polynomial multiplied by $\binom{n+2}{2}^6$ at $c = 1$ and is homogeneous of degree $6n$. Choose $s = 1$ if $3 \mid n$ and $s = 3$ otherwise. For $a < b$, divide the sth roots of all three eigenvalues by $b^{1/s} - a^{1/s}$. After this positive rescaling their roots have the form $(x, x+1, x+1+t)$ with $x, t \ge 0$. Since $sn/3$ is integral, the substituted expression for L is an integer polynomial in x, t. Exact multiplication over all 26 cases gives a total of 1,670,650 nonzero coefficients, each positive. Homogeneity and continuity cover all eigenvalues, including coincidences. At $n = 12$, the average contains 455 words and the polynomial has 2,324 nonzero coefficients, the smallest 2,912. Replay for degrees 11 through 36 and standard-library degree-twelve replay.

1. IQOQI Vienna, Open Quantum Problems, problem 40, posed by D. Hägele.
2. D. Hägele, On the lower bound of the refined Bessis–Moussa–Villani conjecture, arXiv:2611.00129v1 (2026), Sections 2–5.
3. E. H. Lieb and R. Seiringer, Equivalent forms of the Bessis–Moussa–Villani conjecture (2004). H. R. Stahl, Proof of the BMV conjecture (2013).
4. H. Cha and J. Lee, One-parameter counterexamples to the refined Bessis–Moussa–Villani conjecture (2026), version 6.
5. The Omega Institute, Three-spectrum product and exponential lower bounds, public Lean proof archive (2026).